Multiple Unicast Capacity of 2-Source 2-Sink Networks

Chenwei Wang, Tiangao Gou and Syed A. Jafar
EECS Dept., University of California, Irvine, Irvine, CA 92697
Email: {chenweiw, tgou, syed}@uci.edu

***Abstract*—We study the sum capacity of multiple unicasts in wired and wireless multihop networks. With 2 source nodes and 2 sink nodes, there are a total of 4 independent unicast sessions (messages), one from each source to each sink node (this setting is also known as an X network). For wired networks with arbitrary connectivity, the sum capacity is achieved simply by routing. For wireless networks, we explore the degrees of freedom (DoF) of multihop X networks with a layered structure, allowing arbitrary number of hops, and arbitrary connectivity within each hop. For the case when there are no more than two relay nodes in each layer, the DoF can only take values $1, \frac{4}{3}, \frac{3}{2}$ or 2, based on the connectivity of the network, for almost all values of channel coefficients. When there are arbitrary number of relays in each layer, the DoF can also take the value $\frac{5}{3}$. Achievability schemes incorporate linear forwarding, interference alignment and aligned interference neutralization principles. Information theoretic converse arguments specialized for the connectivity of the network are constructed based on the intuition from linear dimension counting arguments.**

I. Introduction

Capacity characterization for multiple unicasts is one of the most important problems in network information theory. Optimal interference management principles are essential to the multiple unicast problem, both in the wireless setting where interference among concurrent transmissions is an unavoidable property of the propagation medium, as well as in the wired network setting where inter-session network coding gives rise to interference among multiple flows. The study of multiple unicast networks has produced many powerful ideas that embrace interference – such as network coding and interference alignment – and that have shown that the capacity limits can be much higher than possible with conventional interference avoidance approaches that do not allow the mixing of flows, such as routing for wired networks and TDMA/FDMA for wireless networks. The idea of interference alignment has been applied primarily to *single hop* wireless networks, where it has significantly advanced the understanding of signal dimensions in the form of degrees of freedom (DoF) characterizations. Network coding principles are most well understood in the multicast setting where all messages are desired by all destinations. With a few notable exceptions (including most recently, [1]–[3]), the problem of multiple unicasts over multiple hops remains wide open for both wired and wireless networks. In this work our goal is to make progress on this problem, by characterizing the sum capacity and degrees of freedom of multiple unicasts over wired and wireless layered multihop networks, respectively.

A. Problem Description

Consider a communication network with M distributed source nodes $s_1, s_2, \cdots, s_M$, and N distributed sink nodes $d_1, d_2, \cdots, d_N$. A total of MN independent unicast sessions are possible in this network, one for each source-destination pair. We are interested in the multiple unicast capacity, which we define as the maximum possible sum-rate of all MN unicast sessions as they simultaneously flow through the network. In more standard information-theoretic terminology, we have MN independent messages $W_{mn}, 1 \leq m \leq M, 1 \leq n \leq N$, with messages W_{mn} originating at source s_m and intended for sink d_n, and we wish to find the sum-rate capacity of these messages. Borrowing the corresponding nomenclature from single hop wireless networks [4], [5], we refer to the setting defined above as the $M \times N$ user X network.

Aside from its significance as the original setting for interference alignment [4], [6], an X network is interesting because the sum capacity of an X network measures the total amount of information that can flow through the network between a *set of distributed sources* and a *set of distributed destinations* without restricting the associations between source-destination pairs. Since each source has an independent message for each destination, all paths that go through the network can carry desired information. However, the total amount of information between the set of sources and the set of destinations is, in general, different from the min-cut between the set of sources and the set of destinations, because of the assumption of *distributed* sources and *distributed* destinations, i.e., the sources cannot share messages and destinations cannot jointly process the received signals. For instance, the 2×2 user X network in the single hop wireless setting is shown to have DoF = $4/3$ in [4], while the DoF min-cut outer bound is 2.

As described above, the distributed nature of sources and destinations and the presence of a desired message from each source to each destination are the defining features of the X network setting. The network between the sources and destinations, can be wired or wireless, single or multiple hop. In this work, we will study two different kinds of X networks.

1) Wired X network: We consider this network in the general setting, i.e., we allow any number of source nodes, any number of destination nodes, any number of hops, and arbitrary network graph topologies comprised of orthogonal noise-free links. Our goal is to characterize the sum-capacity.
2) Layered Wireless X network: Such a network is illustrated in Fig.1. As shown in the figure, we restrict

attention to the $M = 2$, $N = 2$ multihop wireless setting with a layered structure, i.e., a multihop wireless X network, with arbitrary number of layers (hops), an arbitrary number of relay nodes in each layer, and arbitrary connectivity within each hop. Because this is the wireless setting, it incorporates both interference and broadcast features of wireless propagation. Our goal is to characterize the sum DoF.

B. Summary of Contribution

The wired X network sum-capacity bears a surprisingly simple solution. *The sum-capacity is equal to the min-cut separating all sources from all destinations, and is achieved simply by routing.* There is no need for interference alignment and there is no need for either intra-session or inter-session network coding. Since the proof is exceedingly simple, we will describe it here.

Suppose we allow all sources to share all messages, and we allow all destinations to share all their received signals. Then we have essentially a single source, single destination network. We know that the min-cut bound is achievable for this network and a routing solution can be found by the Ford Fulkerson algorithm. Since only routing is needed, there is no mixing of information, i.e., there is no need for cooperation among source nodes or among the destination nodes. Thus, the min-cut is also achievable in the wired X network with distributed sources and destinations.

The main focus of this paper is on the layered multihop wireless X network. Here we proceed in two steps. First, for the case that the number of relay nodes in each layer is no more than 2, we provide an explicit enumeration of all possible network connectivity patterns along with their associated DoF characterizations (in the almost surely sense). In particular, we find that the DoF can only take values $1, \frac{4}{3}, \frac{3}{2}, 2$. Next we allow arbitrary number of relays in each layer and show that here, in addition to networks with DoF values $1, \frac{4}{3}, \frac{3}{2}, 2$, there exist networks with DoF $= \frac{5}{3}$. Further, these are the only multiple unicast DoF values possible for all connectivity patterns in a 2-source 2-sink layered multihop wireless network (for almost all values of channel coefficients). In establishing these results, non-trivial achievability arguments make use of the aligned interference neutralization concept introduced earlier in [1]. Non-trivial outer bounds are also needed, e.g., for the DoF = $\frac{5}{3}$ case. The intuition for the information theoretic outer bounds is obtained from linear dimension counting arguments.

It is interesting to contrast the 2×2 X network with the 2 user interference network, since the only difference between the two settings is in the message sets, i.e., both settings can be defined for the same physical network. While the X network has 4 independent messages, the interference network has only 2 independent messages. In the *one-hop wireless* setting, the X network is much more interesting than an interference network from a DoF perspective, because the X network requires interference alignment, whereas orthogonal access is DoF-optimal for the interference network. In the *multi-hop wired* setting, the opposite is true. The interference network is interesting because it creates opportunities for network coding (e.g., the famous butterfly network), but the X network, as explained earlier in this section, achieves sum-capacity through simple orthogonal access (routing), i.e., requiring neither interference alignment nor network coding. Finally, in the *layered multi-hop wireless* setting, as it turns out, neither the interference network, nor the X network setting is trivial. The DoF of the layered multi-hop interference network are characterized in [2] and are shown to only take values $1, 3/2$ and 2. We show here that the layered multi-hop X network setting presents an even richer picture and gives rise to DoF values $1, 4/3, 3/2, 5/3$ and 2. In both cases, both achievability and converse arguments are non-trivial. For instance, the 2 hop layered network with 2 relays makes use of the idea of aligned interference neutralization, originally introduced in [1], whether it is the interference network or the X network. In addition, the layered multihop X network gives rise to other cases where aligned interference neutralization is needed, such as the network with $5/3$ DoF.

II. System Model and Definitions

The multihop wireless X network we consider in this paper consists of two sources s_1, s_2, two destinations d_1, d_2 and multiple relay nodes between sources and destinations. Each node has one antenna. There are a total of four independent messages in this network, i.e., source s_m wants to send the message W_{mn} to the destination d_n where $m, n \in \{1, 2\}$. We can use a directed graph $G = (V, E)$ to characterize the network topology, where V and E are the sets of nodes and edges, respectively. Such an example is shown in Fig.1. The network has a layered structure. Specifically, for a L-hop network, the two sources are at layer 0, the two destinations are at the layer L, and the relay nodes at the l^{th} $(1 \leq l \leq L-1)$ layer can only receive signals sent from the nodes at the $(l-1)^{th}$ layer, and only transmit to nodes at the $(l+1)^{th}$ layer. In other words, in the graph there are only edges between nodes in adjacent layers. With the layered assumption, we consider an arbitrarily connected network, in the sense that in any two adjacent layers, each node at l^{th} layer can be arbitrarily connected to the nodes in $(l+1)^{th}$ layer. We also assume that every relay node belongs to at least one directed path from at least one source to at least one destination, because otherwise it can be removed without decreasing the capacity region of the network.

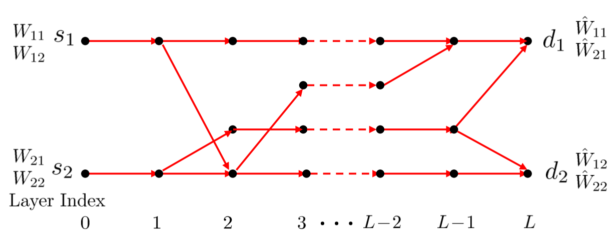

Fig. 1. The Layered Wireless X Network

We denote the i^{th} node in the l^{th} layer as v_i^l. The channel coefficient associated with the edge from the node v_i^l to the node v_j^{l+1} is denoted as $H_{v_j^{l+1} v_i^l}$. We assume that the channel

coefficients are independently drawn from continuous distributions and once drawn, they remain constant during the entire transmission. We also assume that global channel knowledge is available at all nodes. At time index $t \in \mathbb{Z}_+$, each node v_i^l (except the two destinations) transmits a complex-valued signal $X_{v_i^l}(t)$, which satisfies an average power constraint $\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}[|X_{v_i^l}(t)|^2] \leq P$, for T channel uses. The signal received at the node v_j^{l+1} at time t is given by

$$Y_{v_j^{l+1}}(t) = \sum_{v_i^l \in V_j^{l+1}} H_{v_j^{l+1}v_i^l} X_{v_i^l}(t) + Z_{v_j^{l+1}}(t) \qquad (1)$$

where V_j^{l+1} is the set of the nodes connected to v_j^{l+1} at the l^{th} layer, and $Z_{v_j^{l+1}}(t)$ is the i.i.d. additive circularly symmetric complex Gaussian noise with zero-mean unit-variance at the node v_j^{l+1}.

The capacity region $\mathcal{C}(\rho)$ of this network is the set of achievable rate tuples $R(\rho) = (R_{W_{11}}(\rho), R_{W_{12}}(\rho), R_{W_{21}}(\rho), R_{W_{22}}(\rho))$ where ρ is the SNR, such that each user can simultaneously decode its desired messages with arbitrarily small error probability. The maximum sum rate of this channel is defined as $R_{\text{sum}}(\rho) = \max_{R(\rho)\in\mathcal{C}(\rho)} \sum_{m=1}^{2}\sum_{n=1}^{2} R_{W_{mn}}(\rho)$. The capacity in the high SNR regime can be characterized through DoF, i.e., DoF $= \lim_{\rho\to\infty} R_{\text{sum}}(\rho)/\log\rho$. For simplicity, we use d_{mn} to denote the number of DoF associated with the message W_{mn}. Note that we use the notation $o(x)$ to represent any function $f(x)$ such that $\lim_{x\to\infty} f(x)/x = 0$.

III. DoF of 2^{L+1} X Networks

In this section we consider a special class of layered X networks – the 2^{L+1} X network. By 2^{L+1} X network, we mean a layered multihop X network with $L+1$ layers (L hops), and with only two nodes at each layer. Such an example is shown in Fig. 2.

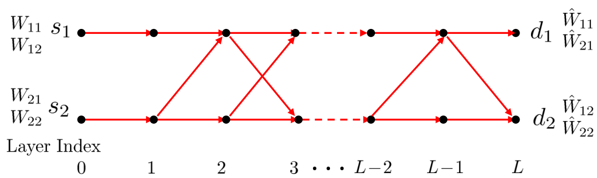

Fig. 2. The 2^{L+1} Wireless X network

Since the DoF min-cut 1 case is trivial, let us consider DoF min-cut 2 networks. Between two adjacent layers, we enumerate all topologies of a one-hop component in Fig. 3. There are

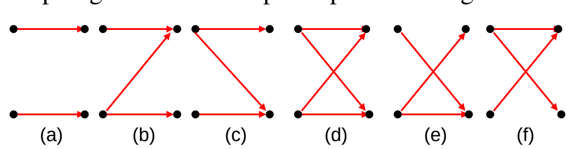

Fig. 3. Components Enumeration of the Wireless X Networks

six cases (a) to (f) depending on the connectivity. Case (a) is trivial because two layers can be collapsed into one. Since the sum capacity of X networks is not affected by switching node labels within a layer, it is clear that subnetworks (e) and (b) are equivalent, and similarly subnetwork (f) is equivalent to (c). Therefore, in the following we only consider the permutations of three components (b), (c), (d). For brevity, we call the three components (b), (c), (d) the "Z", "S" and "X" components, respectively.

We have the following DoF result.

Theorem 1: *For 2^{L+1} X networks defined above, the DoF are given by:*

(A) *DoF $= 1$, if $L = 1$, and the network is a "Z" or "S" network.*
(B) *DoF $= 4/3$, if $L = 1$, and the network is an "X" network.*
(C) *DoF $= 3/2$, if $L \geq 2$, and the network is one of the eight networks: XZ^{L-1}, XS^{L-1}, Z^{L-1}X, S^{L-1}X, ZS^{L-1}, SZ^{L-1}, S^{L-1}Z and Z^{L-1}S.*
(D) *DoF $= 2$, otherwise.*

Proof: Cases (A) and (B) follow from previously known results [4].

Case (C): Since switching node labels within each layer does not affect sum-capacity for the X setting, the eight connectivity patterns for Case (C) can be reduced to the four patterns : XZ^{L-1}, Z^{L-1}X, ZS^{L-1} and S^{L-1}Z. Further, due to the space limitation, we only sketch the argument that shows that the DoF of XZ^{L-1} network is $\frac{3}{2}$. Detailed proofs for all connectivity patterns are presented in the full paper [7].

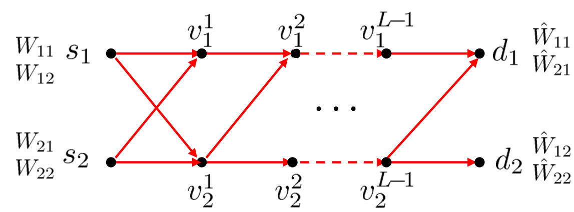

Fig. 4. The XZ^{L-1} Wireless X Network

The $L+1$ layered XZ^{L-1} network is shown in Fig. 4. Let us first consider the DoF outer bound.

DoF Outer Bound: Because each of the source and destination nodes has only one antenna, it is trivial to see that the following four inequalities are satisfied:

$$d_{11} + d_{12} \leq 1, \qquad (2)$$
$$d_{21} + d_{22} \leq 1, \qquad (3)$$
$$d_{11} + d_{21} \leq 1, \qquad (4)$$
$$d_{12} + d_{22} \leq 1. \qquad (5)$$

In Fig. 4, the destination node d_2 can decode its desired messages W_{12} and W_{22}. Since the path from v_2^1 to d_2, i.e., $P_{v_2^1, v_2^2, \cdots, v_2^{L-1}, d_2}$, is free of interference, and the two messages W_{12}, W_{22} must go through this path, every relay node in this path can decode W_{12}, W_{22} as well.

Consider the node v_2^1. Let us set the message $W_{11} = \phi$ to bound the rates for the remaining three messages. Since v_2^1 is able to decode the message W_{12}, after decoding it v_2^1 can remove the signal carrying W_{12} originating from s_1, and then obtain an AWGN channel directly connected to the source s_2. Therefore, subject to the noise distortion which does not affect the number of DoF[1], v_2^1 can also decode the messages W_{21}

[1]We use the phrase "subject to noise distortion" to indicate the widely used (see e.g., [4]) DoF outer bound argument whereby reducing noise at a node by an amount that is SNR independent (and therefore inconsequential for DoF) allows it to decode a message.

and W_{22}. Since single-antenna node v_2^1 is able to decode all the three messages W_{12}, W_{21} and W_{22}, we have the following DoF outer bound:

$$d_{12} + d_{21} + d_{22} \leq 1. \tag{6}$$

Similarly we can set $W_{21} = \phi$. Again since v_2^1 is able to decode whatever d_2 can decode, v_2^1 can decode W_{22} first and then remove the signal carrying it and thus only sees an AWGN channel directly connected to s_1. Subject to the noise distortion v_2^1 can also decode the messages W_{11} and W_{12}, and thus we have another inequality:

$$d_{11} + d_{12} + d_{22} \leq 1. \tag{7}$$

Adding up all inequalities (4), (6) and (7), we have:

$$2(d_{11} + d_{12} + d_{21} + d_{22}) \leq 3. \tag{8}$$

Therefore, we have the outer bound DoF $\leq \frac{3}{2}$.

Achievability: We are going to show a simple scheme that can achieve $3/2$ DoF. We claim that the XZ^{L-1} ($L > 2$) network can also achieve $3/2$ DoF if the two-hop XZ network achieves $3/2$ DoF. Intuitively, this is because by simply repeating (amplify and forward) whatever the intermediate relays from the 2^{nd} to $(L-1)^{th}$ layers receive, we can convert "Z^{L-1}" to one "Z" component. Thus, we only need to prove $3/2$ DoF is achievable for the two-hop XZ network, as shown in Fig.5.

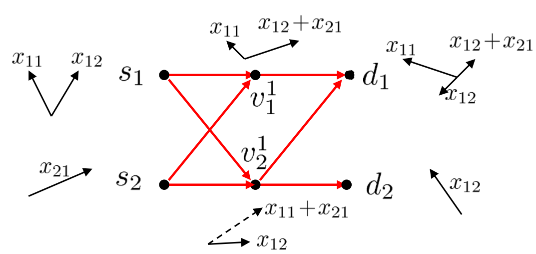

Fig. 5. The "XZ" Wireless X Network

The achievable scheme relies on the idea of aligned interference neutralization. In addition, since the channel is constant, we will use signalling in rational dimensions first introduced in [8], [9]. Over two rational dimensions[2], s_1 sends two symbols x_{11}, x_{12}, and s_2 sends x_{21} each carrying $\frac{1}{2}$ DoF and along a "beamforming" direction. As shown in Fig. 5 we first randomly pick the direction of x_{21} at s_2. Note that although we use vectors to denote the "beamforming" directions in Fig. 5 for simplicity, they should be rationally independent numbers. The direction of x_{11} at s_1 is then fixed by aligning two symbols x_{11} and x_{21} at v_2^1, and the direction of x_{12} at s_1 is fixed by aligning two symbols x_{12} and x_{21} at v_1^1. At the first layer, v_1^1 sends two symbols x_{11} and $x_{12}+x_{21}$, each carried by a randomly picked beamforming direction. The node v_2^1 first demodulates x_{12} and $x_{11}+x_{21}$, and then only sends x_{12} with a beamforming direction such that x_{21} can be canceled at d_1 by the signal coming from v_1^1. Since the directions carrying x_{11} and x_{21} are rationally independent, the destination d_1 is able to decode them, thus achieving one DoF. Also, because v_2^1 only sends x_{12}, and the destination d_2 sees a clean channel, d_2 can decode its desired symbol as well to achieve $1/2$ DoF. Therefore, a total of $3/2$ DoF is achievable.

[2] The concepts of rational dimensions and rational independence, first proposed for real-valued numbers, can be applied to complex-valued numbers as well, as reported in [10]. Also, if the channel is time-varying or frequency-selective, we can also create linear space by using symbol extension, and the rational alignment scheme can be translated to the linear scheme.

Case (D): In this case, we claim that except for the connectivity patterns covered in case (C), all the other channels where $L \geq 2$ have 2 DoF. The DoF outer bound for these networks is trivial and the achievability can be shown based on eliminating two of the 4 messages to reduce the network to an interference channel, so that the results of [2] can be applied. The main observation here, as also in [2], is that in layered multihop networks if interference arrives through more than one path, it can be neutralized.

The eight cases in class (C) have two characteristics: (1) there is only one path from s_1 to d_2 or from s_2 to d_1, and (2) the "Z" or "S" components should be consecutive.

If the first condition does not hold, then it implies that from s_1 to d_2 and from s_2 to d_1, the number of paths is zero or more than one. In this case, by setting $W_{12} = W_{21} = \phi$, either there is no interference, or there are more than 1 paths carrying interference which allows interference neutralization. In either case, 2 DoF are achieved.

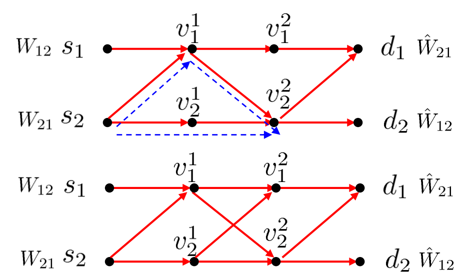

Fig. 6. X Networks with Two DoF

Next, consider the class of connectivity patterns where the second condition is not satisfied but the first condition still holds. In Fig. 6 we show two specific networks characterizing the main properties of this class of networks. Take the first as an example, if we set $W_{11} = W_{22} = \phi$ then it forms a two user interference network, in which it is easy to see that two DoF are achievable even though there is only one path from s_1 to d_2. This is because the message W_{21} intended for the destination d_1 can always be nulled at v_2^2 after going through two paths P_{s_2,v_1^1,v_2^2} and P_{s_2,v_2^1,v_2^2} (denoted by the dashed lines) such that v_2^2 and thus d_2 is interference-free. Since W_{12} can still arrive at d_2 through the path P_{s_1,v_1^1,v_2^2,d_2}, the destination d_2 can achieve one DoF. Similarly, d_1 can also ahieve one DoF. Thus, a total of two DoF is achievable. ■

IV. DoF of General Multihop Layered X Network

So far, we studied the DoF of X networks where there are only two relay nodes at each layer. If the number of relay nodes is not limited to two, one question is whether the DoF of the network with arbitrary connectivity still belong to the set

$\{1, \frac{4}{3}, \frac{3}{2}, 2\}$. Interestingly, there is another class of X networks that have $5/3$ DoF. Due to the space limitation, we will only show one specific example in such a new class. The detailed description and analysis for this new class are reported in [7].

Theorem 2: The network of Fig. 7 has $5/3$ DoF.

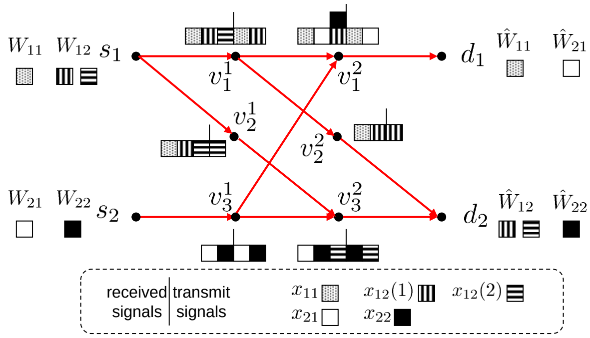

Fig. 7. A Three-hop Layered Network wth $5/3$ DoF

A. DoF Outer Bound

Similar to the analysis in Case (C) of Section III, if we set $W_{12} = \phi$, then we have the following inequality:

$$d_{11} + d_{21} + d_{22} \leq 1. \tag{9}$$

Notice that $Y_{d_1}^n$ only depends on $Y_{v_1^2}^n$. Because d_1 is able to decode W_{11}, v_1^2 can decode it as well. After decoding W_{11}, v_1^2 can subtract the signal $X_{v_1^1}^n$ carrying W_{11}, thus obtaining $X_{v_3^1}^n$ subject to the noise distortion which depends on the channel coefficients but is independent of SNR, and thus has only an $o(\log(\text{SNR}))$ impact on the rate. Since v_3^1 can decode W_{21} and W_{22}, v_1^2 can decode W_{21}, W_{22} as well. Thus, the single-antenna node v_1^2 can decode all these three messages, which implies (9).

Next we derive a new information-theoretical DoF outer bound. Consider the sum rate of two messages desired at the destination d_1. A genie provides W_{22} to the node v_1^2.

$$\begin{aligned}
& n(R_{W_{21}} + R_{W_{11}}) \\
& \leq I(W_{21}, W_{11}; Y_{d_1}^n) + o(n) && (10)\\
& \leq I(W_{21}, W_{11}; Y_{v_1^2}^n) + o(n) && (11)\\
& \leq I(W_{21}, W_{11}; Y_{v_1^2}^n, W_{22}) + o(n) && (12)\\
& \leq I(W_{21}, W_{11}; W_{22}) + I(W_{21}, W_{11}; Y_{v_1^2}^n | W_{22}) + o(n) && (13)\\
& = I(W_{21}, W_{11}; Y_{v_1^2}^n | W_{22}) + o(n) && (14)\\
& = h(Y_{v_1^2}^n | W_{22}) - h(Y_{v_1^2}^n | W_{21}, W_{11}, W_{22}) + o(n) && (15)\\
& \leq n(\log \rho) - h(Y_{v_1^2}^n | W_{21}, W_{11}, W_{22}) + o(n) && (16)\\
& = n(\log \rho) - h(Y_{v_2^2}^n | W_{21}, W_{11}, W_{22}) + n\ o(\log \rho) + o(n). && (17)
\end{aligned}$$

Here, (10) follows from Fano's inequality. (11) follows from the data processing inequality, because $(W_{11}, W_{21}) - Y_{v_1^2}^n - Y_{d_1}^n$ forms a Markov chain. (12) is obtained because providing genie does not decrease the capacity region of the network. (13) follows from the chain rule. (14) is obtained since W_{22} is independent of (W_{21}, W_{11}). (17) follows from the invertibility of the channels (regardless of the values of the channel coefficients as long as they are all non-zero), which implies that given (W_{21}, W_{11}, W_{22}) the entropy of $Y_{v_2^2}^n$ is equal to that of $Y_{v_1^2}^n$ subject to the noise distortion. Specifically, knowing (W_{21}, W_{22}) we can reconstruct the signal $X_{v_3^1}^n$, and by subtracting it from $Y_{v_1^2}^n$ we obtain the signal $X_{v_1^1}^n$ subject to the noise distortion which will depends on the channel coefficients but is independent of SNR. The entropy of $Y_{v_2^2}^n$ is equal to that of $X_{v_1^1}^n$ subject to the noise distortion which again depends only on the channel coefficients but is independent of SNR. All these operations only have an $o(\log(\text{SNR}))$ impact on rate, and so we obtain $h(Y_{v_1^2}^n | W_{21}, W_{11}, W_{22}) = h(Y_{v_2^2}^n | W_{21}, W_{11}, W_{22}) + n\ o(\log \rho)$ as shown in (17). By rearranging terms of (17) we obtain the first outer bound:

$$\begin{aligned}
& n(R_{W_{21}} + R_{W_{11}}) + h(Y_{v_2^2}^n | W_{21}, W_{11}, W_{22}) \leq \\
& \qquad n(\log \rho) + n\ o(\log \rho) + o(n). && (18)
\end{aligned}$$

Next, let us consider the sum rate of two messages originating from the source s_2. A genie provides W_{11} to the nodes v_2^2 and v_3^2.

$$\begin{aligned}
& n(R_{W_{21}} + R_{W_{22}}) \\
& \leq I(W_{21}, W_{22}; Y_{d_1}^n, Y_{d_2}^n) + o(n) && (19)\\
& \leq I(W_{21}, W_{22}; Y_{v_1^2}^n, Y_{v_2^2}^n, Y_{v_3^2}^n) + o(n) && (20)\\
& \leq I(W_{21}, W_{22}; Y_{v_1^2}^n, Y_{v_2^2}^n, Y_{v_3^2}^n, W_{11}) + o(n) && (21)\\
& = I(W_{21}, W_{22}; Y_{v_2^2}^n, Y_{v_3^2}^n, W_{11}) \\
& \quad + I(W_{21}, W_{22}; Y_{v_1^2}^n | Y_{v_2^2}^n, Y_{v_3^2}^n, W_{11}) + o(n) && (22)\\
& \leq I(W_{21}, W_{22}; Y_{v_2^2}^n, Y_{v_3^2}^n, W_{11}) + n\ o(\log \rho) + o(n) && (23)\\
& = I(W_{21}, W_{22}; Y_{v_3^2}^n | Y_{v_2^2}^n, W_{11}) \\
& \quad + I(W_{21}, W_{22}; Y_{v_2^2}^n, W_{11}) + n\ o(\log \rho) + o(n) && (24)\\
& = I(W_{21}, W_{22}; Y_{v_3^2}^n | Y_{v_2^2}^n, W_{11}) + n\ o(\log \rho) + o(n) && (25)\\
& \leq h(Y_{v_3^2}^n | Y_{v_2^2}^n, W_{11}) + n\ o(\log \rho) + o(n) \\
& \quad - h(Y_{v_3^2}^n | Y_{v_2^2}^n, W_{11}, W_{21}, W_{22}) && (26)\\
& \leq n(\log \rho) - h(Y_{v_3^2}^n | Y_{v_2^2}^n, W_{11}, W_{21}, W_{22}) \\
& \quad + n\ o(\log \rho) + o(n) && (27)
\end{aligned}$$

where (19) follows from Fano's inequality. (20) follows from the data processing inequality because $(W_{21}, W_{22}) - Y_{v_1^2}^n - Y_{d_1}^n$, and $(W_{21}, W_{22}) - (Y_{v_2^2}^n, Y_{v_3^2}^n) - Y_{d_2}^n$ form two Markov chains. (23) follows from the invertibility of channels which implies that using $(Y_{v_2^2}^n, Y_{v_3^2}^n, W_{11})$ we can recover the signal $Y_{v_1^2}^n$ subject to the noise distortion. Specifically, we can use $(Y_{v_2^2}^n, Y_{v_3^2}^n)$ to decode W_{12} because $Y_{d_2}^n$ only depends on $(Y_{v_2^2}^n, Y_{v_3^2}^n)$. Thus, using W_{11} and W_{12} we can recover the signal $X_{s_1}^n$ subject to the noise distortion. By knowing $X_{s_1}^n$ we also know $(X_{v_1^1}^n, X_{v_2^1}^n)$. Thus, given $(X_{v_2^1}^n, Y_{v_3^2}^n)$, we can reconstruct the signal $X_{v_3^1}^n$ subject to the noise distortion. Finally, given $(X_{v_3^1}^n, X_{v_1^1}^n)$ we can reconstruct the signal $Y_{v_1^2}^n$. All these operations only have an $o(\log(\text{SNR}))$ impact on rate, so we obtain $I(W_{21}, W_{22}; Y_{v_1^2}^n | Y_{v_2^2}^n, Y_{v_3^2}^n, W_{11}) \leq n\ o(\log \rho) + o(n)$ as in (23). (25) is obtained since (W_{21}, W_{22}) are independent of $(Y_{v_2^2}^n, W_{11})$. (26) follows from the chain rule. By rearranging terms of (27) we obtain the second outer bound:

$$\begin{aligned}
& n(R_{W_{21}} + R_{W_{22}}) + h(Y_{v_3^2}^n | Y_{v_2^2}^n, W_{11}, W_{21}, W_{22}) \leq \\
& \qquad n(\log \rho) + n\ o(\log \rho) + o(n). && (28)
\end{aligned}$$

In the following, we consider the rate of message W_{12}. We provide genie (W_{11}, W_{21}, W_{22}) to the nodes (v_2^2, v_3^2).

$$
\begin{aligned}
nR_{W_{12}} &\leq I(W_{12}; Y_{d_2}^n) + o(n) && (29)\\
&\leq I(W_{12}; Y_{v_2^2}^n, Y_{v_3^2}^n) + o(n) && (30)\\
&\leq I(W_{12}; Y_{v_2^2}^n, Y_{v_3^2}^n, W_{11}, W_{21}, W_{22}) + o(n) && (31)\\
&\leq I(W_{12}; W_{11}, W_{21}, W_{22}) + o(n) \\
&\quad + I(W_{12}; Y_{v_2^2}^n, Y_{v_3^2}^n | W_{11}, W_{21}, W_{22}) && (32)\\
&= h(Y_{v_2^2}^n, Y_{v_3^2}^n | W_{11}, W_{21}, W_{22}) + o(n) \\
&\quad - h(Y_{v_2^2}^n, Y_{v_3^2}^n | W_{11}, W_{12}, W_{21}, W_{22}) && (33)\\
&\leq h(Y_{v_2^2}^n, Y_{v_3^2}^n | W_{11}, W_{21}, W_{22}) + n\ o(\log\rho) + o(n) && (34)\\
&\leq h(Y_{v_2^2}^n | W_{11}, W_{21}, W_{22}) + n\ o(\log\rho) + o(n) \\
&\quad + h(Y_{v_3^2}^n | Y_{v_2^2}^n, W_{11}, W_{21}, W_{22}) && (35)
\end{aligned}
$$

where (29) follows from Fano's inequality. (34) is obtained because knowing the four messages $(W_{11}, W_{12}, W_{21}, W_{22})$ we can reconstruct the signals $(Y_{v_2^2}^n, Y_{v_3^2}^n)$ subject to the noise distortion.

Adding up inequalities (18), (28) and (35), we have:

$$n(2R_{W_{21}} + R_{W_{11}} + R_{W_{22}} + R_{W_{12}}) \leq 2n(\log\rho) + n\ o(\log\rho) + o(n). \quad (36)$$

Dividing $n(\log\rho)$ on both sides of (36), and taking $n \to \infty$, $\rho \to \infty$, we obtain the following inequality:

$$2d_{21} + d_{11} + d_{22} + d_{12} \leq 2. \quad (37)$$

Now adding up inequalities (2), (5), (9) and (37), we obtain:

$$3(d_{11} + d_{12} + d_{21} + d_{22}) \leq 5. \quad (38)$$

Thus, the total DoF of this network is bounded above by $5/3$.

B. Achievability of $5/3$ *DoF*

We provide an interference alignment scheme that can achieve $5/3$ DoF in the network in Fig.7. Over three rational dimensions, source s_1 sends one symbol x_{11} to d_1, two symbols $x_{12}(1),\ x_{12}(2)$ to d_2, and s_2 sends one symbol x_{21} to d_1, and one symbol x_{22} to d_2, each carrying $\frac{1}{3}$ DoF along a rationally independent "beamforming" direction. For brevity, in Fig. 7 we also use boxes (each box denotes one symbol, carrying $1/3$ DoF) with different patterns, to show how our scheme works. We consider the transmission schemes from each layer to the next in what follows.

From layer 0 to layer 1: The source s_1 randomly picks three rationally independent beamforming directions to transmit $x_{11},\ x_{12}(1),\ x_{12}(2)$. Because the edges P_{s_1,v_1^1} and P_{s_1,v_2^1} are AWGN channels, v_1^1 and v_2^1 both can decode these three symbols. Similarly, v_3^1 can decode $x_{21},\ x_{22}$.

From layer 1 to layer 2: After decoding x_{11}, $x_{12}(1)$ and $x_{12}(2)$, v_1^1 sends x_{11} and $x_{12}(1)$ to v_1^2 and v_2^2 using any two rationally independent beamforming directions $U_{v_1^1}(x_{11})$ and $U_{v_1^1}(x_{12}(1))$, respectively. v_2^1 only sends $x_{12}(2)$ to v_3^2 with a randomly picked beamforming direction. v_3^1 sends x_{21} with another randomly picked beamforming direction, but x_{22} in the direction $U_{v_3^1}(x_{22})$ such that x_{22} aligns with $x_{12}(1)$ at v_1^2 in the same dimension, i.e., $H_{v_1^2 v_3^1} U_{v_3^1}(x_{22}) = H_{v_1^2 v_1^1} U_{v_1^1}(x_{12}(1))$.

From layer 2 to layer 3: The node v_1^2 can see a three-rational dimensional space, each dimension carrying symbols x_{11}, x_{21} and $x_{12}(1) + x_{22}$, respectively. Thus, it can demodulate these symbols and only transmits x_{11}, x_{21} to the destination d_1 such that d_1 achieves $2/3$ DoF. v_2^2 receives two symbols x_{11} and $x_{12}(1)$ in two dimensions, and thus it can demodulate them and only sends $x_{12}(1)$ to the destination d_2 with an randomly picked beamforming direction. Similarly, v_3^2 receives three symbols $x_{21},\ x_{22},\ x_{12}(2)$ in three rationally independent dimensions. Thus, it can demodulate them and only transmits $x_{22},\ x_{12}(2)$ to d_2 with two randomly picked rationally independent beamforming directions. At the destination d_2, because it receives three desired symbols in three rationally independent dimensions, it can achieve 1 DoF.

Therefore, a total of $5/3$ DoF is achievable almost surely.

Since both outer and inner bounds are $\frac{5}{3}$ DoF, we establish that the network has a total of $5/3$ DoF. ■

V. Conclusion

Total degrees of freedom (DoF) for multiple unicasts over 2 source 2 sink layered multihop wireless networks are shown to take values $1, 4/3, 3/2, 2$, depending on the connectivity within each hop, for almost all values of channel coefficients, when the number of relayed in each layer is no more than 2. If the number of relays at each layer is not restricted to 2, it is shown through an example, that the network can also have DoF value $5/3$. Finally, we are able to show in [7] that 1, 4/3, 3/2, 5/3 and 2 (in the almost surely sense) are the only possible DoF values for *all* connectivity patterns.

Acknowledgement

The authors would like to acknowledge prior collaboration with Hamed Maleki at UC Irvine and Prof. Sriram Vishwanath at UT Austin for the wired X network capacity result.

References

[1] T. Gou, S. Jafar, S. Jeon, S. Chung, "Aligned Interference Neutralization and the Degrees of Freedom of the $2 \times 2 \times 2$ Interference Channel", *e-print arXiv:1012.2350*, Dec. 2010.

[2] I. Shomorony and S. Avestimehr, "Two unicast wireless networks: characterizing the degrees-of-freedom", *arXiv:1102.2498*, Mar. 2011.

[3] K. Cai, K. B. Letaief, P. Fan, R. Feng, "On the Solvability of 2-pair Unicast Networks — A Cut-based Characterization", *arXiv:1007.0465v1*, July 2010.

[4] S. Jafar, S. Shamai, "Degrees of Freedom Region for the MIMO X Channel", *IEEE Transactions on Information Theory*, vol. 54, No. 1, pp. 151-170, Jan. 2008.

[5] V. Cadambe and S. Jafar, "Interference alignment and the degrees of freedom of wireless X networks", *IEEE Trans. on Information Theory*, vol. 55, no. 9, pp. 3893–3908, Sep. 2009.

[6] M.A. Maddah-Ali, A.S. Motahari , and A.K. Khandani, "Communication Over MIMO X Channels: Interference Alignment, Decomposition, and Performance Analysis," IEEE Transaction on Information Theory, Vol. 54, No. 8, pp. 3457-3470, Aug. 2008.

[7] C. Wang, T. Gou and S. Jafar, "Degrees of Freedom of Layered Multihop X Networks", *Full paper in preparation.*

[8] R. Etkin and E. Ordentlich, "The Degrees of Freedom of the K User Gaussian Interference Channel Is Discontinuous at Rational Channel Coefficients", *IEEE Trans. on Infor. Theory*, vol. 55, no. 11, Nov. 2009.

[9] A.S. Motahari, S. O. Gharan and A. K. Khandani, "Real Interference Alignment with Real Numbers," *arXiv:0908.1208*, Aug. 2009.

[10] M.A. Maddah-Ali, "On the Degrees of Freedom of the Compound MIMO Broadcast Channels with Finite States", *arXiv:0909.5006v3*, Oct. 2009.